

APPLICATION OF FIRST ORDER DIFFERENTIAL EQUATION

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1. Abstract : This Paper, then gives a brief overview of First order differential equation, its formation and its how to solve it. It also explain how first order differential equation can be used to predict growth of bacteria. It also explains how it can be used by archaeological department to estimate the age of fossil. It also thrown light on application of First order differential equation by Police Department to estimate the time of death of a murdered person. In this paper, population of Haryana, a prosperous state of India is predicted by use of First Order equation.

Key Words : First order differential equation, order, degree of differential equation, Formation of differential equations.

2. Introduction : First order Differential Equation is a relation between independent variable dependent variable. derivative of dependent variable in which all its derivatives occur in first degree only and no product of dependent variable and its derivatives occur. For example

$F\left(x, y, \frac{dy}{dx}\right) = 0$ is general form of a differential equation of First order involving one independent and one dependent variable.

2.1 Formation of differential Equation of First order : It is formed by eliminating the arbitrary constants in differential equation.

For example : Let $f(x, y, \square) = 0$ where \square is arbitrary constant.

Differentiating if w.r.t. x , we get a relation

$$g(x, y, \frac{dy}{dx}, \square) = 0 \dots\dots\dots(2)$$

Eliminating \square between (1) and (2) we get a relation

$$F(x, y, \frac{dy}{dx}) = 0 \dots\dots\dots(3)$$

2.2 Solution of differential Equation of First order :

One of such methods is of variable separable which is mostly used.

Let $f(x) dx + g(y) dy = 0 \dots\dots\dots(1)$ is an differential equation

So, $f(x) dx = -g(y) dy \dots\dots\dots(2)$

Integrating both sides, we get

$$\int f(x) dx = -\int g(y) dy + c \dots\dots\dots(3)$$

For example a simple diff. Equ. is

$$(x + y) dy + (1 + y) dx = 0 \quad \dots\dots (1)$$

It can be written as

$$\frac{dy}{1+y} + \frac{dx}{1+x} = 0 \quad \dots\dots (2)$$

Integrating, we get

$$\log(1+y) = -\log(1+x) + \log c \quad \dots\dots(3)$$

$$(1+y) = c(1+x)^{-1} \quad \dots\dots(4)$$

$$(1+x)(1+y) = c \quad \dots\dots(5)$$

3. Application of First order differential Equations :

In various theories involving either growth or decay mostly occurring, equation is $\frac{dv}{dt} = KN$; $N(t_0) = N_0$ where k is a constant.

For example : in biology it is observed that the rate at which certain bacteria grow is proportional to the number of bacteria present at any time over short intervals of time. The population of small animals such as rats can also be predicted by solution of this equation. The population of a city/state can also be predicted by using this initial value problem.

3.1 Problem I : Bacteria in a certain culture increase at a rate proportional to the number present. If the number increase from 1000 to 2000 in 1 hour. How many bacteria are present at the end of 2 hour.

Let N is the number of bacteria present at time t .

So, differential equation is

$$\frac{dN}{dt} = kN \quad \dots\dots(1)$$

$$\frac{dN}{N} = k dt$$

Integrating both sides we get

$$\log N = kt + c$$

$$N = e^{kt} \cdot e^c = N_0 e^{kt} \quad \dots\dots(2)$$

When $t = 0$, $N = 1000$

So putting in (2) we get

$$1000 = e^{0k} \cdot N_0 = N_0 = 1000$$

So, putting in equation (2) we get

$$N = 1000 e^{kt} \quad \dots\dots(3)$$

When $t = 1$ hour, $N = 2000$

So, From equation (3), we get

$$2000 = 1000 e^{k \cdot 1}$$

$$\Rightarrow e^k = 2$$

When $t = 2$ hours, we get

$$N = 1000 e^{2k}$$

$$= 1000 (2)^2$$

$$= 4000$$

So, at the end of 2 hours, there are 4000 bacteria

3.2 Problem 2 :

Archaeological department discovered a specimen whose 20 percent of original radiocarbon has been decomposed. Estimate the number of years since the specimen was a part of living tree.

We know uranium 238 undergoes radioactive decay with half life $T = 4.55$ billion years. During decay it becomes radium 226 and eventually ends as non radioactive lead 206. In theory of radioactivity this knowledge is used to estimate the date of events that took place long ago. The age of solar system has been estimated by radioactive dating as 4.5 billion years. Discovery of carbon-14 by American Scientist. Williard Frank has been valuable in this field. Cosmic rays are continuously entering in earth's atmosphere and produce neutrons which combine with nitrogen to produce ^{14}C . This radio carbon ^{14}C mixed with carbon dioxide and moves through the atmosphere to be absorbed by plants and when animals eat plants, radiocarbon is built in animal tissues. In living tissues, the rate of ingestion of ^{14}C exactly balances the rate of disintegration of ^{14}C . When an organism dies, it ceases to ingest ^{14}C , its concentration begins to decrease through disintegration of ^{14}C present.

In this problem let N is amount of carbon ^{14}C present at time t

$$N = N_0 e^{-kt} \quad \dots\dots(1)$$

where N_0 is amount of carbon present initially at $t = 0$

$$\text{Now } 0.80 N_0 = N_0 e^{-kt}$$

$$\text{So } e^{-kt} = 0.80 \quad \dots\dots(2)$$

$$\Rightarrow kt = \log 0.80$$

Now half life of ^{14}C is 5730 years

Putting in equation (1), we get

$$\frac{N_0}{2} = N_0 e^{-5730 k}$$

$$\Rightarrow \frac{1}{2} = e^{-5730 k}$$

Taking logarithm of both sides, we get

$$-\log 2 = 5730 k$$

$$\Rightarrow k = \frac{-\log 2}{5730}$$

Putting in equation (2), we get

$$\frac{-\log 2}{5730} t = \log 0.80$$

$$\Rightarrow t = \frac{-\log 0.80}{\log 2} \times 5730 = 1844 \text{ years}$$

Problem 3.3 : Archaeological department discovered a fossil which was found to contain $\frac{1}{100}$

of its original amount of ^{14}C . We will determine its age.

Let N is amount of ^{14}C present at time t and N_0 at time $t = 0$

Then,

$$N = N_0 e^{kt}$$

$$\text{When } t = 5730 \text{ years } N = \frac{N_0}{2}$$

$$\text{So, } \frac{N_0}{2} = N_0 e^{kt} = N_0 e^{5730k}$$

$$\Rightarrow \frac{1}{2} = e^{5730k}$$

$$\Rightarrow -\log 2 = 5730k$$

$$\Rightarrow k = \frac{-\log 2}{5730} = -0.0000525305$$

$$\text{When, } N = \frac{1}{100} N_0$$

$$\text{We have } \frac{1}{100} N_0 = N_0 e^{-0.0000525305t}$$

$$\Rightarrow \frac{1}{100} = e^{-0.0000525305t}$$

here t comes out to be equal to 38073 years.

Problem 3.4 :

Police department also uses first order differential equation to determine the time of death of a murdered person.

For example police recovered body of a murder victim at 10 p.m. Doctor took the temperature of body at 10:30 pm which was 93.6°F . He again took the temperature after one hour when it showed 92.4°F . He noticed the temperature of room was 70°F . Now police determined the exact time of death in the following way.

The differential equation in this situation is

$$\frac{dT}{dt} = k(T - T_0)$$

$$\Rightarrow \frac{dT}{dt} - kT = -kT_0$$

$$\Rightarrow (D - k)T = -kT_0 \text{ where } D = \frac{d}{dt}$$

Auxillary equation is $D - k = 0$

$$\Rightarrow D = k$$

$$\text{So C.F.} = C e^{kt}$$

$$\text{P.I.} = \frac{1}{D - k} - kT_0 = \frac{1}{D - k} - kT_0 e^{ot}$$

$$= \frac{1}{0-k} - kT_0 = T_0$$

So solution is

$$T = T_0 + Ce^{kt} \quad \dots\dots\dots(1)$$

When $t = 0$, $T = 93.6$

$$93.6 = T_0 + C \text{ and } T_0 = 70$$

$$93.6 = 70 + C$$

$$C = 23.6$$

When $t = 60$ minutes, $T = 92.4^\circ\text{F}$,

$$\text{So, } 92.4 = 70 + C e^{60k} \quad [\text{From equation (i)}]$$

$$\Rightarrow e^{60k} = \frac{22.4}{23.6}$$

$$\Rightarrow k = \frac{1}{60} \log \frac{22.4}{23.6}$$

$$\Rightarrow -0.0003781333 \quad \dots\dots\dots(2)$$

Now $T = T_0 + C e^{kt}$

$$98.4 = 70 + 23.6 e^{-0.0003781333t} \quad [\text{using equation 2}]$$

$$28.4 = 23.6 e^{-0.0003781333t}$$

$$\text{So, } \frac{28.4}{23.6} = e^{-0.0003781333t}$$

$$\Rightarrow t = -212.62 \text{ minute}$$

$$= -3.54 \text{ hours.}$$

So, Time of death = 10 – 3 hours 54 minute

$$= 6.06 \text{ pm}$$

Problem 3.5 :

Population of Haryana in 2011 was 2.54 crore in 2012. Its population was 2.58 crore. Calculate its population after 10 years starting from 2011

Let $N(t)$ – population at time t

$N(0)$ - initial population

$$N(t) = N(0) e^{kt}$$

When $t = 0$, $N(t) = 2.54$ crore

$$\text{So, } 2.54 = N(0) e^{k \cdot 0}$$

$$\Rightarrow N(0) = 2.54$$

$$\text{So, } N(t) = 2.54 e^{kt}$$

When $t = 1$,

$$N(t) = 2.58 \text{ crore}$$

So, we get

$$2.58 = 2.54 e^{k \cdot 1}$$

$$\Rightarrow e^k = \frac{2.58}{2.54} = \frac{258}{254}$$

When, $t = 10$, $N(t)$ will be given by

$$N(t) = 2.54 e^{10k}$$

$$= 2.54 \left(\frac{258}{254} \right)^{10}$$

$$\Rightarrow 2.96926 \text{ crore}$$

This will be population after 10 years

Problem 3.6 : In orthogonal trajectory first order differential equation is also employed to find orthogonal trajectory.

For example, a charged particle may moving under the influence of a magnetic field always travels on a curve which is perpendicular to each of magnetic field lines. The orthogonal trajectory of a family of curves can be obtained as follows :

Let $F(x, y, \square) = 0$ (1) is a family of curves

differentiating (1) w.r.t x , we get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} \quad \dots\dots(2)$$

Now we substitute value of \square from equation (1) into equation (2) since slopes of curves which intersect

orthogonally are negative reciprocals of each other. So orthogonal trajectory of family of curve of (1) are

given by

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For example

We find the orthogonal trajectory of family of parabolas

$$y^2 = cx \quad \dots\dots(1)$$

Differentiating (1) w.r.t x , we get

$$2y \frac{dy}{dx} = c \quad \dots\dots(2)$$

$$\text{From (1) } c = \frac{y^2}{x}$$

$$\text{So, } 2y \frac{dy}{dx} = \frac{y^2}{x}$$

$$\Rightarrow \frac{y^2}{x} = \frac{y}{2x}$$

Since sloper of curve which intersect (1) orthogonally are negative reciprocals of each other.

So, orthogonal trajection will be given by

$$\frac{dy}{dx} = -\frac{2x}{y}$$

$$\text{So, } ydy + 2x dx = 0$$

Integrating, we get

$$\frac{y^2}{2} + x^2 = c$$
$$\Rightarrow \frac{x^2}{1} + \frac{y^2}{2} = c$$

Which is a family of ellipse

1. **Conclusion :** It is incontrovertible fact that the differential Equations form the most important branch of modern mathematics and in fact, occupy the central position in both pure and applied mathematics. So far as the pure mathematics is concerned attempts to get exact solutions of differential equation give way to Existence theorems and the Theorem of functions. Differential equations form the basis of applied Mathematics. My present paper dealt with some applications of First order differential equations which include estimating age of a fossil, checking crime. estimating population of Haryana a State of India etc.

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